PROBLEMS OF OPTIMIZING MULTILAYERED SPHERICAL VESSELS

Yu. V. Nemirovskii and M. L. Kheinloo

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The problem of calculating a composite construction possessing defined optimal properties when compared with other constructions of the given type is considered in this paper in the case of spherical vessels in the domain of elastic strains.

1. Let a spherical vessel constitute a set of N concentric spheres connected with one another (Fig. 1) and made from materials which in the general case are different. The vessel is loaded by uniform internal and external pressures and is under the conditions of stationary heating.

We introduce the dimensionless quantities

$$\begin{gathered}\psi_i = \psi_i^\circ / pl, \quad r_i = \rho_i / l, \quad T_i = T_i^\circ / T_*,\\ \beta_i = \beta_i^\circ T_* \quad E_i = E_i^\circ / p, \quad \sigma_{\theta_i} = \sigma_{\theta_i}^\circ / p,\\ \sigma_{r_i} = \sigma_{r_i}^\circ / p, \quad \nu_i, \ \alpha_i = a_i / l, \alpha_{N+1} = b / l,\\ h_i = h_i^\circ / l, \quad p_i = p_i^\circ / p \quad u_i = u_i^\circ / l,\\ \sigma_i^* = \sigma_i^{*\circ} / p, \quad p_{N+1} = \overset{\circ}{p}_{N+1} / p\end{gathered}$$

where p, T_*, l are the characteristic pressure, temperature, and linear dimension; ψ_i° and T_i° are functions of the stress and temperature in the i-th (i = 1, 2, 3, ..., N) sphere; E_i°, ν_i, β_i° are Young's modulus, Poisson's ratio, and the coefficient of linear expansion in the i-th sphere; $\sigma^\circ_{r_i}, \sigma^\circ_{\theta i} = \sigma^\circ_{\varphi i}$, u_i° are the stress and displacement components in the i-th sphere; a_i and h_i° are the inside radius and thickness of the i-th sphere; p_i° is the reactive pressure on the inner surface of the i-th sphere; b and p°_{N+1} are the outside radius and the outside pressure for the composite spherical vessel; ρ_i ($a_i \le \rho_i \le a_{i+1}$) is the current radius within the limits of the i-th sphere; and $\sigma_i^{*\circ}$ is the limiting stress in the i-th sphere.

We shall assume in the general case that Young's moduli vary with temperature according to some law $E_i = E_i(T_i)$. Then for a certain sphere with the number i (Fig. 1) (the numeration begins from the inner cavity) we obtain the equation [1, 2]

$$\psi_i'' + \left[\frac{2}{r_i} - \frac{E_i'}{E_i}\right]\psi_i' - \left[\frac{E_i'(1-3\nu_i)}{E_i r_i (1-\nu_i)} + \frac{2}{r_i^2}\right]\psi_i = -\frac{2\beta_i}{(1-\nu_i)} T_i'(r_i) E_i$$

(a prime denotes a derivative with respect to the dimensionless coordinate r_i); the general solution of this equation has the form

$$\psi_i = \sum_{j=1}^{3} c_{ji}\psi_{ji}, \quad c_{3i} = 1 \quad (i = 1, 2, \ldots, N)$$

Here ψ_{1i} and ψ_{2i} are the particular solutions of the homogeneous equation, while ψ_{3i} is the particular solution of the nonhomogeneous equation.

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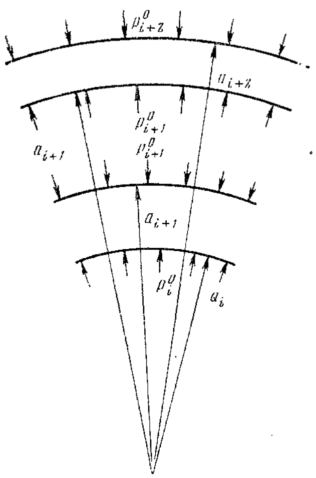

Fig. 1

If the material of all spheres is incompressible, then for any relationship $E_i = E_i\,(T_i)$ we have

$$\psi_{1i} = r_i, \qquad \psi_{2i} = r_i \int g_i dr_i, \qquad g_i = r_i^{-4} E_i$$

$$\psi_{3i} = -4\beta_i r_i \int g_i \left(\int f_i dr_i\right) dr_i, \qquad f_i = T_i' r_i^3$$

If the material of the spheres is compressible and Young's modulus does not depend on temperature, then

$$\psi_{1i} = r_i, \qquad \psi_{2i} = r_i^{-2}, \qquad \psi_{3i} = -2\beta_i E_i r_i^{-2} (1 - \nu_i)^{-1} \int T_i r_i^2 dr_i$$

and in the absence of a temperature field

$$\psi_{1i} = r_i, \qquad \psi_{2i} = r_i^{-2}, \qquad \psi_{3i} = 0 \tag{1.1}$$

The stresses σ_{r_i}, σ_{i_θ} and the displacements u_i are determined in terms of the stress functions ψ_i by means of the expressions

$$\sigma_{r_i} = \sum_{j=1}^{3} c_{ji} \psi_{ji} r_i^{-1}, \qquad \sigma_{\theta_i} = 2^{-1} \sum_{j=1}^{3} c_{ji} (\psi_{ji} r_i^{-1} + \psi_{ji}') \tag{1.2}$$

$$u_i = r_i E_i^{-1} \sum_{j=1}^{3} A_{ji} c_{ji} + \beta_i r_i T_i, \qquad A_{ji} = 2^{-1} [(1 - 3\nu_i)\psi_{ji} r_i^{-1} + (1 - \nu_i)\psi_{ji}']$$

where the constants of integration c_{1i} and c_{2i} are determined from the boundary conditions

$$\sigma_{r_i}(\alpha_i) = -p_i, \qquad \sigma_{r_N}(\alpha_{N+1}) = -p_{N+1}$$

and are equal to

$$c_{ji} = (-1)^{j+1} (D_{ji} + B_{ji}) D_i^{-1}$$
$$D_i = \psi_{2i}(\alpha_i)\psi_{1i}(\alpha_{i+1}) - \psi_{2i}(\alpha_{i+1})\psi_{1i}(\alpha_i)$$
$$D_{ji} = \alpha_i p_i \psi_{ni}(\alpha_{i+1}) - \alpha_{i+1} p_{i+1} \psi_{ni}(\alpha_i)$$
$$B_{ji} = \psi_{3i}(\alpha_i)\psi_{ni}(\alpha_{i+1}) - \psi_{3i}(\alpha_{i+1})\psi_{ni}(\alpha_i), \quad n = 1, 2, \quad n \neq j$$

In a number of cases it is expedient to design multilayered vessels which are strengthened by fairly thin spherical shells made from high-strength materials. If a layer with the number i corresponds to such a shell, then in (1.2) we must put

$$\sigma_{r_i} = 0, \quad \sigma_{\theta_i} = \sigma_i = \alpha_i (p_i - p_{i+1})(2h_i)^{-1}$$
$$u_i = \alpha_i^2 (p_i - p_{i+1})(1 - \nu_i)(2E_i h_i)^{-1} + \alpha_i \beta_i T_i(\alpha_i) \tag{1.3}$$

Indeed, using the Lagrange theorem ([3], page 128) and the condition $h_i \ll \alpha_i$ (and, in fact, $\alpha_i \approx \alpha_{i+1}$, $\psi_{ji}(\alpha_i) \approx \psi_{ji}(\alpha_{i+1})$, $\psi_{ji}'(\alpha_i) \approx \psi_{ji}'(\alpha_{i+1})$), we have

$$D_i = [\psi_{2i}(\alpha_i)\psi_{1i}'(\alpha_i) - \psi_{2i}'(\alpha_i)\psi_{1i}(\alpha_i)]\, h_i$$
$$D_{ji} = \alpha_i \psi_{ni}(\alpha_i)(p_i - p_{i+1}), \quad n = 1, 2, \quad n \neq j$$
$$B_{ji} = [\psi_{3i}(\alpha_i)\psi_{ni}'(\alpha_i) - \psi_{3i}'(\alpha_i)\psi_{ni}(\alpha_i)]\, h_i$$

Substituting these expressions into (1.2), we obtain the expressions (1.3). Using the joining conditions

$$u_k(\alpha_{k+1}) = u_{k+1}(\alpha_{k+1}) \qquad (k = 1, 2, \ldots, N-1)$$

we obtain the system of equations for the determination of the reactive forces p_j ($j = 2, 3, \ldots, N$):

$$d_{1k}p_k + p_{k+1} + d_{2k}p_{k+2} = M_k \tag{1.4}$$

where

$$d_{1k} = H_{1k}S_k^{-1}, \quad d_{2k} = H_{2k}S_k^{-1}, \quad M_k = R_kS_k^{-1} \tag{1.5}$$

$$H_{1k} = \alpha_k\,[A_{1k}(\alpha_{k+1})\,\psi_{2k}(\alpha_{k+1}) - A_{2k}(\alpha_{k+1})\,\psi_{1k}(\alpha_{k+1})]\,D_{k+1}$$

$$H_{2k} = \alpha_{k+2}\,[A_{1,k+1}(\alpha_{k+1})\,\psi_{2,k+1}(\alpha_{k+1}) - A_{2,k+1}(\alpha_{k+1})\,\psi_{1,k+1}(\alpha_{k+1})]\,D_kn_k$$

$$R_k = D_kn_kL_{k+1}(\alpha_{k+1}) - L_k(\alpha_{k+1})\,D_{k+1}$$

$$L_k(r_k) = B_{1k}A_{1k}(r_k) - B_{2k}A_{2k}(r_k) + D_kA_{3k}(r_k) + \beta_kT_k(r_k)\,D_kE_k$$

$$S_k = \alpha_{k+1}\,\{D_{k+1}\,[A_{2k}(\alpha_{k+1})\,\psi_{1k}(\alpha_k) - A_{1k}(\alpha_{k+1})\,\psi_{2k}(\alpha_k)] + n_kD_k\,[A_{2,k+1}(\alpha_{k+1})\,\psi_{1,k+1}(\alpha_{k+2}) -$$

$$- A_{1,k+1}(\alpha_{k+1})\,\psi_{2,k+1}(\alpha_{k+2})]\}, \quad n_k = E_kE_{k+1}^{-1}$$

The general solution of the system (1.4) is obtained by the Cramer method [4]

$$p_{k+1} = \Delta_k\Delta^{-1} \qquad (k = 1, 2, \ldots, N-1) \tag{1.6}$$

where

$$\Delta = \begin{vmatrix} 1 & d_{21} & 0 & 0 & \ldots & 0 & 0 \\ d_{12} & 1 & d_{22} & 0 & \ldots & 0 & 0 \\ 0 & d_{13} & 1 & d_{23} & \ldots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \ldots & d_{1,N-1} & 1 \end{vmatrix}$$

while Δ_k is obtained by replacing the k-th column by the column of the right side of the system (1.4).

We denote

$$J_1 = 1, \quad J_t = 1 - d_{1t}d_{2,t-1}J_{t-1}^{-1} \qquad (t = 2, 3, \ldots, N-1) \tag{1.7}$$

Since $J_k \neq 0$ [for $\Delta = J_1J_2 \ldots J_{N-1} \neq 0$, since the system (1.4) must have a unique solution], we obtain the following solution of the system (1.4), using the Gauss method [4]:

$$p_2 = K_1J_1^{-1} - K_2d_{21}(J_1J_2)^{-1} + K_3d_{21}d_{22}(J_1J_2J_3)^{-1} - \ldots + (-1)^N K_{N-1}d_{21}d_{22}\ldots d_{2,N-2}(J_1J_2J_3\ldots J_{N-1})^{-1}$$

$$p_3 = K_2J_2^{-1} - K_3d_{22}(J_2J_3)^{-1} + K_4d_{22}d_{23}(J_2J_3J_4)^{-1} - \ldots + (-1)^{N-1}K_{N-1}d_{22}d_{23}\ldots d_{2,N-2}(J_2J_3\ldots J_{N-1})^{-1} \tag{1.8}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$p_{N-1} = K_{N-2}J_{N-2}^{-1} - K_{N-1}d_{2,N-2}(J_{N-1}J_{N-2})^{-1}$$

$$p_N = K_{N-1}J_{N-1}^{-1}$$

$$K_1 = M_1 - d_{11}p_1, \quad K_2 = M_2 - d_{12}[M_1 - d_{11}p_1]\,J_1^{-1}$$

$$K_3 = M_3 - M_2d_{13}J_2^{-1} + d_{12}d_{13}[M_1 - d_{11}p_1]\,(J_2J_1)^{-1}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$K_{N-1} = M_{N-1} - d_{2,N-1}p_{N+1} - M_{N-2}d_{1,N-1}J_{N-2}^{-1} + \ldots +$$

$$+ (-1)^N d_{1,N-1}d_{1,N-2}\ldots d_{13}d_{12}[M_1 - d_{11}p_1]\,(J_{N-2}J_{N-3}\ldots J_3J_2J_1)^{-1}$$

These relationships are convenient since they give explicit expressions of the reactive pressure in terms of the parameters of the composite vessel, external load, and temperature.

2. The general analytical expressions obtained in Section 1 for the determination of the reactive pressures and stress and displacement distributions in the case of an arbitrary assembly of a composite sphere

allows us to formulate and solve a number of problems concerned with optimization of multilayered spherical vessels.

The first fundamental optimization problem is: find such a construction of a composite spherical vessel for which, under given loading (including a temperature field) in all spheres or in some of them, a certain limiting state is fulfilled. For example, at certain points the condition of yield or ultimate strength is satisfied. If an internal thin-walled shell is included in the assembled construction, then loss of stability is possible for a certain critical stress.

The second fundamental optimization problem is: find such a construction of a multilayered spherical vessel that the absolute displacements on the inner or outer surfaces are a minimum.

In addition, in certain constructions it is possible to satisfy both requirements at the same time (the mixed optimization problem).

We introduce the function $f_i(r_i)$ which under the Tresca or Mises plasticity conditions has the form

$$f_i(r_i) = \sigma_{\theta_i} - \sigma_{r_i} = 2^{-1} \sum_{j=1}^{3} c_{ji} [\psi_{ji}'(r_i) - \psi_{ji}(r_i) r_i^{-1}]$$

To solve the first optimization problem we must first of all find $|f_i(r_i)| = |f_i(r_i^*)|$ in the region $\alpha_i \leq r_i \leq \alpha_{i+1}$. Further, the following cases are possible:

1) in spheres with certain indices q the yield conditions are fulfilled at the radii r_q^*; the rest of the spheres remain elastic;

2) the inside sphere does not lose stability [5];

3) all spheres remain elastic;

4) the inside sphere loses stability.

The conditions

$$|f_q(r_q^*)| = \sigma_q^* \qquad (2.1)$$

$$|f_i(r_i)| < \sigma_i^*, \quad r_i \neq r_q^* \quad (i = 1, 2, \ldots, N) \qquad (2.2)$$

$$p_2 - p_1 < p^*, \quad p^* = 2E_1 h_1^2 \alpha_1^{-2} [3(1 - \nu_1^2)]^{-1/2} \qquad (2.3)$$

correspond to the cases 1) and 2).

The inequality (2.2) for all r_i and

$$p_2 - p_1 = p^* \qquad (2.4)$$

correspond to the cases 3) and 4).

Equations (2.1), (2.4) and the inequality (2.2), which should be regarded as constraints on the parameters α_i, E_i, ν_i, σ_i^*, p_1, p_{N+1}, with

$$0 < \alpha_1 \leqslant \alpha_2 \leqslant \ldots \leqslant \alpha_{N+1}$$

correspond to the cases 1) and 4).

In (2.1)-(2.4) the pressures p_j are calculated according to the expressions (1.8).

We note that in the absence of a temperature field

$$f_i(r_i) = 3(p_i - p_{i+1}) \alpha_i^3 \alpha_{i+1}^3 2^{-1} (\alpha_{i+1}^3 - \alpha_i^3)^{-1} r_i^{-3}$$

$$\text{and } \max |f_i(r_i)| = |f_i(\alpha_i)| \quad \text{for} \quad \alpha_i \leqslant r_i \leqslant \alpha_{i+1}.$$

Consequently, in this case in (2.1), (2.2) we must take $r_q^* = \alpha_q$. In the second fundamental optimization problem we can find the minimum of $|u_1|$ or $|u_N|$ with respect to the parameters α_i, ν_i, E_i, p_1, p_{N+1} under the conditions

$$0 < \alpha_1 \leqslant \alpha_2 \leqslant \ldots \leqslant \alpha_{N+1}$$
$$|f_i(r_i)| < \sigma_i^*, \quad p_2 - p_1 < p^* \quad (i = 1, 2, \ldots, N)$$

The solution of this problem on the basis of the general expression (1.2) is carried out by a standard method ([3], page 318). Therefore, further details in this direction are omitted here.

The solution of the systems of nonlinear algebraic equations (2.1) for the parameters being varied can be sought by means of the successive approximations of Newton [6]. Here the problem concerned with the choice of the first approximation must be considered in each particular case separately, in conformity with the choice of the parameters to be varied. For example, when varying the quantity α_j, in the role of the first approximation we must take $\alpha_j = \alpha_1 + (j-1)\ (\alpha_{N+1} - \alpha_1)N^{-1}$.

In a number of cases the specific features of the design of the required composite vessel allows us to simplify, to some degree, the expressions (1.5), (2.1)-(2.4). Let, for example, in a certain part of a multi-layered spherical vessel, thin shells and thick spheres be alternating.

We denote these shells by the indices t, while the spheres alternating with them are denoted by the indices s. Then, using the natural assumptions

$$h_t \ll h_s \tag{2.5}$$

(h_t and h_s denote respectively the thicknesses of the shells and spheres) and stipulating that the conditions of plasticity be fulfilled in certain shells with the indices q (q assumes all or a part of the numbers corresponding to the indices t), instead of Eqs. (2.1) we, for example, obtain

$$|\alpha_q (p_q - p_{q+1}) (2h_q)^{-1}| = \sigma_q^* \tag{2.6}$$

where the solution of this system of equations must satisfy the inequalities (2.2) for $i \neq q$. In addition, on the basis of (2.5) and the Lagrange theorem on a mean value ([3], page 128) the coefficients (1.5) for $k = t$ and $k = s$ respectively assume the form

$$d_{1t} = -(1 + \theta_t h_t)^{-1}, \quad d_{2t} = -\varkappa_t h_t (1 + \theta_t h_t)^{-1} \tag{2.7}$$
$$d_{1s} = -\varkappa_s h_{s+1} (1 + \theta_s h_{s+1})^{-1}, \quad d_{2s} = -(1 + \theta_s h_{s+1})^{-1}$$
$$M_t = N_t h_t (1 + \theta_t h_t)^{-1}, \quad M_s = N_s h_{s+1} (1 + \theta_s h_{s+1})^{-1}$$

$$\theta_t = n_t D_t^* [A_{2,t+1}(\alpha_t)\, \psi_{1,t+1}(\alpha_{t+2}) - A_{1,t+1}(\alpha_t)\, \psi_{2,t+1}(\alpha_{t+2})]\, \varepsilon_t^{-1}$$
$$\varepsilon_t = \overset{**}{D}_{t+1} [A_{2t}(\alpha_t)\, \psi_{1t}(\alpha_t) - A_{1t}(\alpha_t)\, \psi_{2t}(\alpha_t)]$$
$$\varkappa_t = n_t D_t^* \alpha_{t+2} [A_{2,t+1}(\alpha_t)\, \psi_{1,t+1}(\alpha_t) - A_{1,t+1}(\alpha_t)\, \psi_{2,t+1}(\alpha_t)]\, (\alpha_t \varepsilon_t)^{-1}$$
$$N_t = L_t^* \{\alpha_t [A_{2t}(\alpha_t)\, \psi_{1t}(\alpha_t) - A_{1t}(\alpha_t)\, \psi_{2t}(\alpha_t)]\, \overset{**}{D}_{t+1}\}^{-1} \tag{2.8}$$
$$\theta_s = \overset{\circ\circ}{D}_{s+1} [A_{2s}(\alpha_{s+2})\, \psi_{1s}(\alpha_s) - A_{1s}(\alpha_{s+2})\, \psi_{2s}(\alpha_s)]\, \varepsilon_s^{-1}$$
$$\varepsilon_s = D_s^\circ n_s [A_{2,s+1}(\alpha_{s+2})\, \psi_{1,s+1}(\alpha_{s+2}) - A_{1,s+1}(\alpha_{s+2})\, \psi_{2,s+1}(\alpha_{s+2})]$$
$$\varkappa_s = \alpha_s [A_{2s}(\alpha_{s+2})\, \psi_{1s}(\alpha_{s+2}) - A_{1s}(\alpha_{s+2})\, \psi_{2s}(\alpha_{s+2})]\, \overset{\circ\circ}{D}_{s+1} (\alpha_{s+2} \varepsilon_s)^{-1}$$
$$N_s = L_s^\circ \{\alpha_{s+2} D_s^\circ n_s [A_{2,s+1}(\alpha_{s+2})\, \psi_{1,s+1}(\alpha_{s+2}) - A_{1,s+1}(\alpha_{s+2})\, \psi_{2,s+1}(\alpha_{s+2})]\}^{-1}$$
$$B_{jt}^* = \psi_{3t}(\alpha_t)\, \psi_{nt}'(\alpha_t) - \psi_{3t}'(\alpha_t)\, \psi_{nt}(\alpha_t), \quad n = 1, 2, \quad j \neq n$$
$$D_t^* = \psi_{2t}(\alpha_t)\, \psi_{1t}'(\alpha_t) - \psi_{2t}'(\alpha_t)\, \psi_{1t}(\alpha_t)$$
$$\overset{*}{B}_{j,t+1} = \psi_{3,t+1}(\alpha_t)\, \psi_{n,t+1}(\alpha_{t+2}) - \psi_{3,t+1}(\alpha_{t+2})\, \psi_{n,t+1}(\alpha_t), \quad n = 1, 2, \quad j \neq n$$
$$\overset{**}{D}_{t+1} = \psi_{2,t+1}(\alpha_t)\, \psi_{1,t+1}(\alpha_{t+2}) - \psi_{2,t+1}(\alpha_{t+2})\, \psi_{1,t+1}(\alpha_t)$$
$$L_t = D_t^* n_t [\overset{**}{B}_{1,t+1} A_{1,t+1}(\alpha_t) - \overset{**}{B}_{2,t+1} A_{2,t+1}(\alpha_t) + \overset{**}{D}_{t+1} A_{3,t+1}(\alpha_t) + \beta_{t+1} T_t(\alpha_t) \times$$
$$\times \overset{**}{D}_{t+1} E_{t+1}] - \overset{**}{D}_{t+1} [B_{1t}^* A_{1t}(\alpha_t) - B_{2t}^* A_{2t}(\alpha_t) + D_t^* A_{3t}(\alpha_t) + \beta_t T_t(\alpha_t)\, D_t^* E_t$$
$$B_{js}^\circ = \psi_{3s}(\alpha_s)\, \psi_{ns}(\alpha_{s+2}) - \psi_{3s}(\alpha_{s+2})\, \psi_{ns}(\alpha_s), \quad n = 1, 2, \quad j \neq n$$
$$D_s^\circ = \psi_{2s}(\alpha_s)\, \psi_{1s}(\alpha_{s+2}) - \psi_{2s}(\alpha_{s+2})\, \psi_{1s}(\alpha_s)$$
$$\overset{\circ\circ}{B}_{j,s+1} = \psi_{3,s+1}(\alpha_{s+2})\, \psi_{n,s+1}'(\alpha_{s+2}) - \psi_{3,s+1}'(\alpha_{s+2})\, \psi_{n,s+1}(\alpha_{s+2}), \quad n = 1, 2, \quad j \neq n$$
$$\overset{\circ\circ}{D}_{s+1} = \psi_{2,s+1}(\alpha_{s+2})\, \psi_{1,s+2}'(\alpha_{s+2}) - \psi_{2,s+1}'(\alpha_{s+2})\, \psi_{1,s+1}(\alpha_{s+2})$$
$$L_s^\circ = D_s^\circ n_s [\overset{\circ\circ}{B}_{1,s+1} A_{1,s+1}(\alpha_{s+2}) - \overset{\circ\circ}{B}_{2,s+1} A_{2,s+1}(\alpha_{s+2}) + \overset{\circ\circ}{D}_{s+1} A_{3,s+1}(\alpha_{s+2}) +$$
$$+ \beta_{s+1} T_s(\alpha_{s+2})\, \overset{\circ\circ}{D}_{s+1} E_{s+1}] - \overset{\circ\circ}{D}_{s+1} [B_{1s}^\circ A_{1s}(\alpha_{s+2}) - B_{2s}^\circ A_{2s}(\alpha_{s+2}) +$$
$$+ D_s^\circ A_{3s}(\alpha_{s+2}) + \beta_s T_s(\alpha_{s+2})\, D_s^\circ E_s]$$

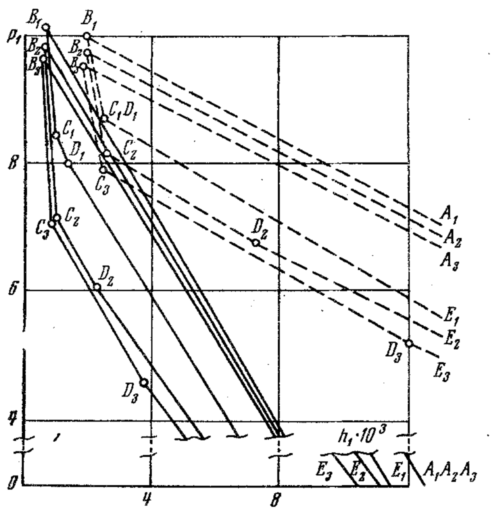

Fig. 2

In these expressions primes denote a derivative with respect to the argument indicated in brackets attached to the function; α_t and α_s are the inside radii of the shells and spheres, where it is assumed that $\alpha_t = \alpha_{t+1}$, $\alpha_{s+1} = \alpha_{s+2}$.

On the basis of (1.7), (2.5), and (2.7) for multilayered spherical vessels with alternating shells it is natural to take

$$J_s = 1, \qquad J_t = 1 - d_{1t} d_{2t,-1} \tag{2.9}$$

Here we must bear in mind that $d_{20} = 0$, i.e., $J_1 = 1$.

In certain cases Eqs. (2.9) allow us to obtain analytical expressions for optimal thickness of the shells.

We consider, for example, the case where the composite vessel with alternating shells is loaded only by the external pressure ($M_1 = M_2 = \ldots = M_{N-1} = p_1 = 0$). Substituting in this case the expressions (1.8) into (2.6), where the absolute-value signs are replaced by the multiplier $k_q = \pm 1$ (plus is taken for tension, minus is taken for compression), with (2.7) and (2.9) taken into account we obtain a system of equations from which by subsequent computations we find

$$\begin{aligned} h_N &= -(2k_N \sigma_N{}^*)^{-1} - \theta_{N-1}^{-1} \quad \text{at} \quad q = N \\ h_q &= \varkappa_q \sigma_{q+2}^* \alpha_q' k_{q+2} (\alpha_{q+2} \theta_{q+1} \sigma_q{}^* \theta_q k_q)^{-1} - \theta_{q-1}^{-1} - \theta_q^{-1} \\ h_1 &= \varkappa_1 \sigma_3{}^* \alpha_1 k_3 (\theta_1 \sigma_1{}^* \alpha_3 k_1)^{-1} - \theta_1{}^{-1} \quad \text{at} \quad q = 1 \end{aligned} \tag{2.10}$$

Here the quantities θ_t, $\varkappa_t$, θ_s, $\varkappa_s$ are obtained from (2.8) when substituting Eqs. (1.1), while the index q corresponds to shells in which Eqs. (2.6) are fulfilled.

The expressions (2.10) give the optimal thicknesses of the shells with the indices q of any composite spherical vessel, in the portion of it where shells alternate with spheres, the vessel being loaded only by external pressure.

Since all h_q, θ_s, $\varkappa_s$, θ_t, $\varkappa_t$ must be positive according to their meaning, we must take $k_q = -1$ in (2.10). Consequently, all shells of an optimal composite construction simultaneously reach the yield points in compression. The solution (2.10) thus obtained must satisfy the inequalities (2.2) for $i \neq q$. These inequalities can always be satisfied at the expense of the choice of the remaining parameters p_1, p_{N+1}, E_i, ν_i, σ_i^* (for $i \neq q$).

3. In the role of an example we consider the problem of a three-layered spherical vessel of the shell-sphere-shell type, loaded by an internal pressure p_1 and an external pressure p_4. In this case N = 3; t = 1, 3; s = 2; q = 1, 2, 3. The inequalities (2.2) and (2.3) determine the region of elastic stress states, while preserving the stability of the inside shell. The boundaries of this region are determined by Eqs. (2.1) and (2.4). For the parameter values

$$\begin{array}{llll} E_1 = 40000, & E_2 = 3000, & E_3 = 14000, & \nu_1 = 0.3 \\ \nu_2 = 0.25, & \nu_3 = 0.3, & \sigma_1{}^* = 100, & \sigma_2{}^* = 1 \\ p_4 = 10.0, & \sigma_3{}^* = 20, & p = |\sigma_2{}^{*\circ}|, & l = 6 \end{array} \tag{3.1}$$

the regions of elastic states are found inside the five-cornered figures $A_kB_kC_kD_kE_k$ shown in Fig. 2.

At the same time the values k = 1, 2, 3 correspond to the indices $h_3 = 0, 0.005, 0.01$. The solid five-cornered figures correspond to the parameter $\alpha_1 = 0.2$; the dashed figures correspond to the parameter $\alpha_1 = 0.6$. The curves A_kB_k and C_kD_k are obtained from Eq. (2.1) for q = 2, and they correspond to the reaching of the yield points on the inside diameter of the sphere, for tension and compression respectively. The curves B_kC_k are determined from Eq. (2.4), and they correspond to the loss of stability of the inside sphere. The curves D_kE_k are determined from Eq. (2.1) for q = 1, and they correspond to the reaching of the yield point on the inside diameter of the inner shell under the conditions of compression. When calculating these curves, no assumpitons were used for the shells. A calculation according to the expressions (2.6) (for q = 1, 3), simplified on the basis of the assumptions (2.5), points to a sufficient reliability of these expressions

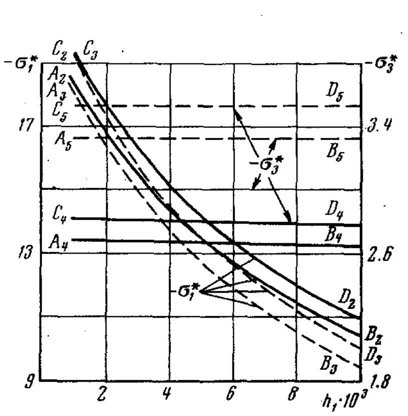

Fig. 3

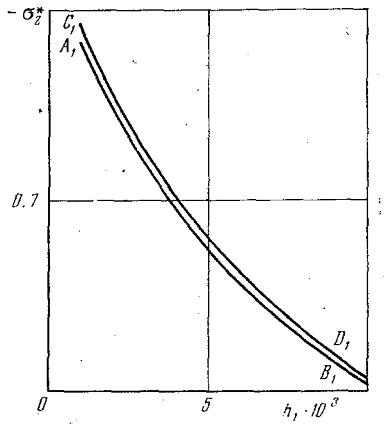

Fig. 4

for finding the curves A_kB_k, B_kC_k, C_kD_k within the whole range of the computed values of h_1 and h_3; for the curves D_kE_k this applies for $h_1 < 7 \cdot 10^{-3}$. The corresponding curves are located fairly close to the curves A_kB_k, B_kC_k, C_kD_k, D_kE_k in Fig. 2 with a maximum deviation of 15% and are not presented because of their cumbersomeness.

We note that in Fig. 2 the points B_k, C_k, and D_k correspond to optimal designs, with the points B_k, C_k corresponding to the yield point in the sphere and the loss of stability of inner shell being reached simultaneously. The points D_k correspond to the yield points being reached simultaneously in the sphere and the inner shell. For the parameter values (3.1) the yield point in the outer shell can be reached only after the yield points in the sphere and (or) the inner shell or after the loss of stability of the inner shell.

The use of graphs like those depicted in Fig. 2, although giving fairly clear information about the state of a composite construction, turns out to be inconvenient for finding optimal parameters of the construction with a large number of elements. In this case it is more convenient to use directly Eqs. (2.1)-(2.4) or the simplified Eq. (2.6). In Figs. 3 and 4 at $p = p_4$, $l = b$, for the parameter values

$$\nu_1 = 0.3,\ \nu_2 = 0.25,\ \nu_3 = 0.3,\ p_1 = 0,\ n_1 = {}^{40}/_3,\ n_2 = {}^{3}/_{14},\ \alpha_1 = 0.2$$

we have presented the graphs of the relationship between $-\sigma_n^*$ (n = 1, 2, 3) and h_1, calculated according to the expressions (2.1) (solid lines) and (2.6) (dashed lines). The curves A_kB_k and C_kD_k (k = 1, 2, 3, 4, 5) are calculated respectively for $h_3 = 0.01$ and 0.001. As is seen from these curves, the use of the assumptions (2.5) is completely justified.

The graphs represented in Figs. 3 and 4 allow us to determine the approximate values of the optimal parameters of a composite spherical vessel. Thus, for example, for

$$\alpha_1 = 0.2,\ n_1 = {}^{40}/_3,\ n_2 = {}^{3}/_{14},\ l = b,\ -\sigma_2^* = 0.27,\ -\sigma_1^* = 16.2, -\sigma_3^* = 2.70,\ p_4^\circ = 185 \ \text{kg/cm}^2$$

from the graphs (solid lines) in Figs. 3 and 4 we see that the feasible values of the optimal parameters must be located in a region which is a common part of the regions

$$h_1 \in 10^{-5}\,[80, 82],\quad h_1 \in 10^{-4}\,[25, 31],\quad h_1 \in 10^{-3}\,[1, 10]$$
$$h_3 \in 10^{-3}\,[1, 10],\quad h_3 \in 10^{-3}\,[1, 10],\quad h_3 \in 10^{-3}\,[1, 10]$$

i.e.,

$$h_1 \in 10^{-5}\,[80, 82],\ h_3 \in 10^{-3}\,[1, 10]$$

Assuming that the curves A_kB_k, when h_3 varies uniformly, are transformed uniformly into the curves C_kD_k, we obtain

$$h_1 \approx 0.0081,\ h_3 \approx 0.0050 \tag{3.2}$$

for the case considered.

For these values the plasticity conditions are satisfied simultaneously on the inside diameter of the sphere and in the outer shell. With the aid of (2.2), (2.3) it is easy to see that the inner shell remains elastic and does not lose stability.

We note that the values (3.2) can be used in the role of the first approximation to improve the values of h_1 and h_3 according to Newton's method with use of Eqs. (2.1), when the thickness of the shells is not too small.

Concluding, we make the following observation. To show that the method of calculation presented is true also in the case where the inner shell loses stability, in (2.3) for the sake of simplicity we used the expression of critical load of the individual shell.

In reality, when the stability loss of the inner shell is accompanied by deformation of the layers to which it is joined, the critical load can be different. The corresponding problem can be solved, and the resulting value must be used for the improvement in (2.3). However, this problem will be considered separately.

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